ISRAEL JOURNAL OF MATHEMATICS **129** (2002), 1–20

ON THE COHOMOLOGY OF DRINFEL'D'S p-ADIC SYMMETRIC DOMAIN

BY

GIL ALON AND EHUD DE SHALIT

Institute of Mathematics, The Hebrew University of Jerusalem
Givat Ram, Jerusalem 91904, Israel
e-mail: alongil@math.huji.ac.il, deshalit@math.huji.ac.il

ABSTRACT

There are, by now, three approaches to the de-Rham cohomology of Drinfel'd's p-adic symmetric domain: the original work of Schneider and Stuhler, and more recent work of Iovita and Spiess, and of de Shalit. In the first part of this paper we compare all three approaches and clarify a few points which remained obscure. In the second half we give a short proof of a conjecture of Schneider and Stuhler, previously proven by Iovita and Spiess, on a Hodge-like decomposition of the cohomology of p-adically uniformized varieties.

Let K be a finite extension of $\mathbb{Q}_p$, V a $(d+1)$-dimensional vector space over K, and V^* its dual. Drinfel'd's *p-adic symmetric domain* of dimension d, $\mathfrak{X}$, is the complement in $\mathbb{P}(V^*)$ of the K-rational hyperplanes, regarded as a rigid analytic space over K. We denote by $\mathcal{A} = \mathbb{P}(V_K)$ the set of K-rational hyperplanes. If it is necessary to distinguish between $a \in \mathcal{A}$ and the hyperplane in $\mathbb{P}(V^*)$ defined by the equation $a(x) = 0$, we write H_a for the latter. Thus

$$\mathfrak{X} = \mathbb{P}(V^*) - \bigcup_{a \in \mathcal{A}} H_a. \tag{0.1}$$

The group $G = \mathrm{PGL}(V_K)$ acts on $\mathfrak{X}$ by rigid motions, and therefore acts on its (rigid) de-Rham cohomology $H^{\cdot}_{dR}(\mathfrak{X}/K)$, which becomes an infinite-dimensional representation of G. The graded algebra $H^{\cdot}_{dR}(\mathfrak{X}/K)$ has a natural subalgebra, the algebra of *logarithmic classes*, generated by the classes $[d\log(a/b)]$ of logarithmic

Received March 1, 2001

1-forms. (Note that a/b, for $a, b \in \mathcal{A}$, is a well-defined and nowhere vanishing function on $\mathfrak{X}$.)

In [S-S] Schneider and Stuhler computed $H^{\cdot}_{dR}(\mathfrak{X}/K)$ as a G-module. A different approach was proposed in [dS], using *residues* to identify the cohomology with the space $C^{\cdot}_{har}$ of *harmonic cochains* on the *Bruhat-Tits building* $\mathcal{T}$ of G. It was shown there that the logarithmic classes are dense in $H^{\cdot}_{dR}(\mathfrak{X}/K)$, and their associated harmonic cochains are given by an explicit combinatorial formula. However, the relationship between the results of [dS] and those of Schneider and Stuhler was not determined until now.

Recently [I-S], Iovita and Spiess managed to compute the isomorphism of Schneider and Stuhler on the logarithmic classes, and obtained a third description of the cohomology, which stimulated our present work.

Our goal is to relate the three approaches. We first re-prove the Main Theorem of [I-S] by the methods of [dS]. The derivation is simpler than that of [I-S], which relies on the definitions of [S-S], and is logically independent of it. As a corollary we get the desired comparison between the isomorphisms of [S-S] and [dS]. We shall also clarify the description of $H^k_{dR}(\mathfrak{X}/K)$ by means of *boundary distributions* on a certain component $\mathcal{B}^{\min}_{k-1}$ of $\mathcal{B}_{k-1}$, the space of flags of length k in V_K (which is closely related to the Borel-Serre boundary of $\mathcal{T}$).

An important proper subspace of the cohomology is the space $H^k_{dR}(\mathfrak{X}/K)_{bnd}$ of *bounded* classes (called bounded logarithmic in [I-S]). Under the isomorphism with C^k_{har} it corresponds to the bounded harmonic cochains, and thus contains all the logarithmic classes. We show that two natural Banach norms on this space — the sup norm on the bounded harmonic cochains and the Banach quotient norm on the space of measures on $\mathcal{A}^{k+1}$ modulo degenerate measures, coincide. As explained in [I-S], a nice feature of $H^k_{dR}(\mathfrak{X}/K)_{bnd}$ is that each bounded cohomology class has a canonical closed form representing it. We call the space of these representatives $\Omega^k_{bnd}(\mathfrak{X})$. One may ask whether $H^k_{dR}(\mathfrak{X}/K)_{bnd}$ is a *maximal* subspace in the cohomology which is represented G-equivariantly on the level of differential forms. We do not know the answer to this question. The formula which recovers the form representing a bounded cohomology class from the associated boundary measure may be considered the *p-adic Poisson integral.* For $k = d$ it was written down, in coordinates, by Schneider and Teitelbaum in [S-T].

A key technical tool is Proposition 5 of [S-S], Section 3, whose proof the authors attribute to Deligne, and which we therefore called the Lemma of Deligne-Schneider-Stuhler. It allows one to interpolate between the cochain complexes of two profinite simplicial sets, one leading to a description of $H^k_{dR}(\mathfrak{X}/K)$ in terms of a *quotient* of the space of distributions on $\mathcal{A}^{k+1}$ and the other as a *subspace* of the space of distributions on $\mathcal{B}_{k-1}^{\min}$.

In the final section we review the application that Iovita and Spiess give to their Main Theorem, namely to the cohomology of $\Gamma\backslash\mathfrak{X}$, when Γ is a discrete and cocompact subgroup of G. Recall that by a theorem of Mustafin, this is the rigid analytic space associated to a unique smooth and projectve variety X_Γ over K. The de-Rham cohomology of X_Γ admits, besides the usual Hodge filtration $F_{dR}^{\cdot}$, another filtration $F_\Gamma^{\cdot}$, induced by the covering spectral sequence. The theorem of Iovita and Spiess (formerly a conjecture of Schneider) says that these two filtrations are opposite, hence induce a Hodge-like direct sum decomposition of the cohomology. In [dS], Section 9, we showed that the two spectral sequences in question are the two spectral sequences attached to a certain natural double complex. Here we want to explain how the isomorphism $\Omega_{bnd}^k(\mathfrak{X}) \simeq H_{dR}^k(\mathfrak{X}/K)_{bnd}$ leads to a short new proof of the degeneration of the two spectral sequences, as well as of Schneider's conjecture on the two induced filtrations.

1. The various approaches to the cohomology

1.1. DE-RHAM COHOMOLOGY. We use the notation and the definitions of [dS]. Let K, G and $\mathfrak{X}$ be as in the introduction. Let $\mathcal{T}$ be the Bruhat-Tits building of G, and $r\colon \mathfrak{X} \to |\mathcal{T}|$ the reduction map from $\mathfrak{X}$ to the topological realization of $\mathcal{T}$. Since $\mathfrak{X}$ is a Stein domain (in the sense of Kiehl), its rigid de-Rham cohomology $H_{dR}^k(\mathfrak{X}/K)$ $(0 \le k \le d)$ is the space of closed k-forms modulo the exact ones. We denote by $[\omega]$ the cohomology class of a closed form ω. Recall that $|\mathcal{T}|$ has the structure of a metric space, in which every apartment is a Euclidean space. Fix a vertex v of $\mathcal{T}$, let $B_n(v)$ be the subcomplex of $|\mathcal{T}|$ spanned by the vertices at distance at most n from v, and $\mathfrak{X}_n(v) = r^{-1}(\int(B_n(v)))$. Each $\mathfrak{X}_n(v)$ is a Stein space whose cohomology is finite dimensional, and

$$(1.1) \qquad H_{dR}^k(\mathfrak{X}/K) = \varprojlim H_{dR}^k(\mathfrak{X}_n(v)/K).$$

The inverse limit topology on $H_{dR}^k(\mathfrak{X}/K)$ is then independent of the choice of v, and G acts continuously.

1.2. FIRST DESCRIPTION OF THE COHOMOLOGY (SCHNEIDER-STUHLER). To follow the approach of [S-S] we have to introduce coordinates. Fix the basis $(e_0, \ldots, e_d)$ of V_K, and identify G with $\mathbf{G} = \mathrm{PGL}_{d+1}(K)$ as in the paragraph preceding theorem 8.3 of [dS]: $g \in G$ corresponds to $\mathbf{g} = (a_{ij}) \in \mathbf{G}$ if

$$g^{-1}e_i = \sum_{j=0}^{d} a_{ij}e_j. \tag{1.2}$$

If we put

$$L_0 = \sum_{i=0}^{d} \mathcal{O}_K e_i,$$

$$W_i^0 = \sum_{j=d+i-k}^{d} Ke_j, \quad L_i = \pi L_0 + (W_i^0 \cap L_0) \quad (1 \le i \le k), \tag{1.3}$$

then the standard k-simplex

$$\sigma_0 = (L_0 \supset L_1 \supset \cdots L_k \supset \pi L_0) \tag{1.4}$$

is of minimal type (i.e., $\dim L_i/L_{i+1} = 1$ for $i \ge 1$; $L_{k+1} = \pi L_0$). Let $I = \{1, \ldots, d-k\}$ and write

$$B_I = \mathrm{Stab}_{\mathbf{G}}(\sigma_0), \tag{1.5}$$

$$P_I = \mathrm{Stab}_{\mathbf{G}}(W_1^0 \supset \cdots \supset W_k^0) \tag{1.6}$$

for the standard parahoric and parabolic subgroups of $\mathbf{G}$ "of type I". Compare [S-S], p. 72, and note that our $(e_0, \ldots, e_d)$ are denoted in that paper $(e_1^*, \ldots, e_{d+1}^*)$. According to our conventions $V_K^* = K^{d+1}$ is represented by column vectors, while $V_K = (K^{d+1})^*$ is represented by row vectors, and the contragredient action of G on V_K^* is the usual left action of $\mathbf{G}$ on K^{d+1}. Let

$$C_{SS}^k = \mathrm{Hom}(C_c^\infty(\mathbf{G}/B_I, \mathbb{Z})/R_I, K), \tag{1.7}$$

where R_I is the G-module of relations introduced in [S-S]. The isomorphism of Schneider and Stuhler ([S-S], corollary 17, p. 83) is a certain (non-explicit) isomorphism

$$c^{SS} : H_{dR}^k(\mathfrak{X}/K) \simeq C_{SS}^k. \tag{1.8}$$

1.3. Second description of the cohomology (as harmonic cochains on $\mathcal{T}$). Let C_{har}^k be the space of *harmonic cochains* on $\widehat{\mathcal{T}}_k$, as defined in [dS], Section 3.1. The main theorem of [dS] asserts that there is an isomorphism

$$c\colon H_{dR}^k(\mathfrak{X}/K) \simeq C_{har}^k, \quad [\omega] \mapsto c_\omega. \tag{1.9}$$

It is defined by the theory of residues: $c_\omega(\sigma)$ is the *residue* of ω along the oriented simplex σ. Furthermore, if $S = (a_0, \ldots, a_k) \in \mathcal{A}^{k+1}$ and

$$\omega = \omega(S) = \sum_{i=0}^{k} (-1)^i d\log a_0 \wedge \cdots \wedge \widehat{d\log a_i} \wedge \cdots \wedge d\log a_k, \tag{1.10}$$

then

$$c_\omega(\sigma) = (\sigma, S), \tag{1.11}$$

where $(\sigma, S) \in \{0, \pm 1\}$ is determined according to the following rule. Assume that σ is defined by a lattice-flag as in (1.4) above. Normalize all the a_i to lie in $L_0 - \pi L_0$, and let $\iota_\sigma(a_i)$ denote the last j such that $a_i \in L_j$. Then if the $\iota_\sigma(a_i)$ are all distinct, (σ, S) is the sign of the permutation $i \mapsto \iota_\sigma(a_i)$, and otherwise it is 0.

Let C^k_{bnd} denote the bounded harmonic cochains. In the sup-norm

$$||c|| = \sup_{\sigma \in \widehat{\mathcal{T}}_k} |c(\sigma)| \tag{1.12}$$

it becomes a K-Banach space.

1.4. Third description of the cohomology (Iovita-Spiess). We first need some notation concerning measures and distributions. If X is any locally compact, totally disconnected topological space, we denote by $L(X) = C_c^\infty(X, \mathbb{Z})$ the space of all $\mathbb{Z}$-valued, compactly supported, locally constant functions on X. For any ring (or abelian group) R we call $D(X, R) = \mathrm{Hom}(L(X), R)$ the space of R-valued *distributions* on X. If μ is a distribution and $f \in L(X)$, we denote $\mu(f)$ by $\int f(x) d\mu(x)$. We shall be interested specifically in K-valued distributions. A bounded distribution, which by definition is an element of $M(X, K) = D(X, \mathcal{O}_K) \otimes_{\mathcal{O}_K} K$, is called a *measure*. If μ is a measure, one can integrate against it not only locally constant, but also continuous, compactly supported K-valued functions.

Our third description of the cohomology goes in a direction opposite to the first two. One starts with a K-valued *distribution* μ on $\mathcal{A}^{k+1}$, where from now on we endow $\mathcal{A}$ with the p-adic topology. The *degenerate distributions* $D(\mathcal{A}^{k+1}, K)_{\deg}$ are the μ's annihilating

$$\Lambda'_k = \mathrm{Ker}(\iota^*, \partial^*)\colon L(\mathcal{A}^{k+1}) \to L((\mathcal{A}^{k+1})_0) \oplus L(\mathcal{A}^{k+2}). \tag{1.13}$$

Here $(\mathcal{A}^{k+1})_0$ is the closed subset of all $(a_0, \ldots, a_k)$ which are linearly dependent, and ι^* is the restriction map. The second map, ∂^*, is the usual coboundary homomorphism. The degenerate measures are the measures which are degenerate as distributions.

To μ Iovita and Spiess attach the integral

$$\Phi(\mu) = \int_{\mathcal{A}^{k+1}} [\omega(S)] d\mu(S) \in H^k_{dR}(\mathfrak{X}/K). \tag{1.14}$$

The point behind this definition is that the restriction to each $\mathfrak{X}_n(v)$ of the class of $\omega(S)$ modulo exact forms is locally constant (in S), so the integral makes sense. Moreover, if μ is a measure, then since $S \mapsto \omega(S)$ is continuous, and since continuous functions can be integrated against measures (by Riemann integration),

$$\omega(\mu) = \int_{\mathcal{A}^{k+1}} \omega(S) d\mu(S) \tag{1.15}$$

is well defined, and $\Phi(\mu) = [\omega(\mu)]$.

It is easy to verify that $\Phi(\mu) = 0$ if μ is degenerate. The Main Theorem of [I-S] is that it induces an isomorphism

$$\Phi\colon D(\mathcal{A}^{k+1}, K)/D(\mathcal{A}^{k+1}, K)_{\deg} \simeq H^k_{dR}(\mathfrak{X}/K). \tag{1.16}$$

In more detail, let $Y_\cdot^{(k)}$ be the profinite simplicial set given by

$$Y_r^{(k)} = \left\{ S = (a_0, \ldots, a_r) \in \mathcal{A}^{r+1};\ \dim(\mathrm{Span}(S)) \le k \right\}, \tag{1.17}$$

and let $|Y_\cdot^{(k)}|$ be its topological realization (see [S-S], p. 65). As explained in Section 2 (see 2.7), there is a canonical isomorphism

$$\tilde{H}^{k-1}(|Y_\cdot^{(k)}|, \mathbb{Z}) \simeq \Lambda'_k, \tag{1.18}$$

for the reduced cohomology, and therefore

$$\mathrm{Hom}(\tilde{H}^{k-1}(|Y_\cdot^{(k)}|, \mathbb{Z}), K) \simeq D(\mathcal{A}^{k+1}, K)/D(\mathcal{A}^{k+1}, K)_{\deg}. \tag{1.19}$$

On the other hand, as Schneider and Stuhler show, there is a canonical isomorphism

$$C^k_{SS} \simeq \mathrm{Hom}(\tilde{H}^{k-1}(|Y_\cdot^{(k)}|, \mathbb{Z}), K). \tag{1.20}$$

Combining (1.20), (1.19) and (1.16), we obtain a homomorphism from C^k_{SS} to $H^k_{dR}(\mathfrak{X}/K)$. The Main Theorem of [I-S] identifies it as the inverse of c^{SS}, thereby reducing the proof that Φ is an isomorphism, to the (known) fact that c^{SS} is an isomorphism.

1.5. FOURTH DESCRIPTION OF THE COHOMOLOGY (THROUGH HARMONIC MEASURES ON THE BOUNDARY). In a slightly different notation, this description appears in [S-S] already. Let $\mathcal{B}.$ denote the profinite simplicial set of flags in V_K. Its $k-1$ simplices $(1 \le k \le d)$ consist of flags of length k

$$\mathcal{B}_{k-1} = \left\{ \begin{array}{c} W_1 \supset \cdots \supset W_k;\ W_i \text{ are proper subspaces of } V_K, \\ \text{properly contained in each other} \end{array} \right\}, \tag{1.21}$$

and the boundary map ∂ takes a flag to the alternating sum of the flags which are obtained from it by omittimg one of the subspaces. We view $\mathcal{B}_{k-1}$ as the space of K-points of a certain flag variety, which is compact and totally disconnected in the p-adic topology. It breaks into $\binom{d}{k}$ components, labelled by the *types* of the flags. The *component of minimal type* $\mathcal{B}_{k-1}^{\min}$ consists of flags where $\dim W_i = k+1-i$.

Consider the space of K-valued distributions $\mathcal{D}(\mathcal{B}_{k-1}^{\min}, K)$. A distribution ν on $\mathcal{B}_{k-1}^{\min}$ is called *harmonic* if either $k=1$ and it is of total mass 0, or $k \ge 2$ and $\partial_* \nu = 0$ (note that ∂ extends to a map $\partial_*\colon \mathcal{D}(\mathcal{B}_{k-1}, K) \to \mathcal{D}(\mathcal{B}_{k-2}, K)$ and $\mathcal{D}(\mathcal{B}_{k-1}^{\min}, K) \subset \mathcal{D}(\mathcal{B}_{k-1}, K)$). The harmonic distributions on $\mathcal{B}_{k-1}^{\min}$ will be denoted $\mathcal{D}(\mathcal{B}_{k-1}^{\min}, K)_{har}$. The space of harmonic measures on $\mathcal{B}_{k-1}^{\min}$ is a K-Banach space in a natural way.

The lemma of Deligne-Schneider-Stuhler (Lemma 2.1 below) implies the isomorphism

$$D(\mathcal{A}^{k+1}, K)/D(\mathcal{A}^{k+1}, K)_{\deg} \simeq \mathcal{D}(\mathcal{B}_{k-1}^{\min}, K)_{har}. \tag{1.22}$$

We therefore obtain a fourth description of the cohomology, from the previous one.

1.6. COMPARISON OF THE VARIOUS APPROACHES. We are now ready to state our results. Call a flag of vector spaces

$$F = (W_1 \supset \cdots \supset W_k) \tag{1.23}$$

compatible with an oriented k-simplex $\sigma = (L_0 \supset \cdots \supset L_k \supset \pi L_0)$ if

$$L_i = \pi L_0 + (W_i \cap L_0). \tag{1.24}$$

Denote by $\mathcal{B}(\sigma)$ the open and compact subset in $\mathcal{B}_{k-1}$ of all the flags which are compatible with σ. Notice that if F is compatible with σ, their types coincide, and in particular F is of minimal type if and only if σ is. The $\mathcal{B}(\sigma)$ form a basis for the topology of $\mathcal{B}_{k-1}$ (compare [SS], Section 4, Proposition 8).

THEOREM 1.1: *Let $[\omega] \in H^k_{dR}(\mathfrak{X}/K)$, and let $c_\omega \in C^k_{har}$ be the associated harmonic cochain. Then (see diagram in Section 2.2):*

(i) *There exists a unique harmonic distribution $\nu_\omega \in \mathcal{D}(\mathcal{B}^{\min}_{k-1}, K)_{har}$ satisfying*

$$\nu_\omega(\mathcal{B}(\sigma)) = c_\omega(\sigma) \tag{1.25}$$

for every $\sigma \in \widehat{\mathcal{T}}^{\min}_k$ (oriented k-simplex of minimal type).

(ii) *If $\mu_\omega \in D(\mathcal{A}^{k+1}, K)$ corresponds to ν_ω under the Deligne-Schneider-Stuhler Lemma (1.22), then*

$$\Phi(\mu_\omega) = [\omega]. \tag{1.26}$$

(iii) *Let $\lambda_\sigma \in L(\mathcal{A}^{k+1})$ be the function $\lambda_\sigma(S) = (\sigma, S)$. Then*

$$\mu_\omega(\lambda_\sigma) = c_\omega(\sigma).$$

(iv) *Introduce coordinates as in Section 1.2. Then*

$$c_\omega(\sigma_0) = c^{SS}_\omega(\chi_{B_I}), \tag{1.27}$$

where $\chi_{B_I} \in L(\mathbf{G}/B_I)$ is the characteristic function of the principal coset B_I.

Part (iv) of the theorem implies that for any $g \in G$,

$$c_\omega(g\sigma_0) = c^{SS}_\omega(\chi_{gB_I}). \tag{1.28}$$

Since G acts transitively on oriented k-simplices of a given type, c^{SS}_ω is the restriction of the harmonic cochain c_ω to cells of minimal type.

Contrary to the impression that (i) might have on the reader, it is *not* true that there is a distribution on all of $\mathcal{B}_{k-1}$ satisfying (1.25) for every σ. Equivalently, if $\mathcal{B}(\sigma)$ is the disjoint union of finitely many $\mathcal{B}(\sigma_i)$, unless the type of σ (and of the σ_i) is minimal, it is not true that $(\sigma, S) = \sum(\sigma_i, S)$ for all $S \in \mathcal{A}^{k+1}$. This "breaking of symmetry" among the types is built into the definition of C^k_{har}; see [dS], Section 3. It also manifests itself in the proof of Lemma 2.1 below, where the arguments work only for flags of minimal type.

THEOREM 1.2: *The following are equivalent.*

(i) $c_\omega \in C^k_{bnd}$.

(ii) *ν_ω is a measure.*

(iii) *μ_ω can be taken to be a measure.*

If this happens we call $[\omega]$ a bounded cohomology class. The norm of c_ω, the norm of ν_ω, and the Banach quotient norm of the class of μ_ω, all coincide.

Furthermore, if we denote by Ω^k_{bnd} *the image of (1.15), i.e., the space of all* k*-forms* $\omega(\mu)$ *for* $\mu \in M(\mathcal{A}^{k+1}, K)/M(\mathcal{A}^{k+1}, K)_{\deg}$, *then the map* $\omega \mapsto [\omega]$ *is a* G*-equivariant isomorphism of* Ω^k_{bnd} *onto the space of bounded cohomology classes.*

CONJECTURE 1.3: *If* $\Omega^k_{bnd} \subset \tilde{\Omega} \subset \Omega^k_{cl}$ *is any* G*-invariant subspace of closed* k*-forms which maps injectively into the cohomology (i.e., which intersects the exact forms trivially), then* $\Omega^k_{bnd} = \tilde{\Omega}$.

Perhaps in the last conjecture one has to impose some topological restrictions on $\tilde{\Omega}$. The passage from a *bounded* ν_ω to μ_ω, followed by (1.15), can be thought of as the p-adic Poisson integral. For $k = d$ it coincides with the formulae of [S-T].

1.7. EXAMPLE: $d = k = 1$. Here $\mathcal{T}$ is the famous tree, and $\mathcal{B}^{\min}_{k-1}$ is $\mathbb{P}(V_K)$, the set of *ends* of the tree. For an oriented edge σ, the set $\mathcal{B}(\sigma)$ consists of the ends of all the rays passing through σ. The function (σ, S) on $S = (a_0, a_1) \in \mathcal{A}^2$ gives S the value 1 if the geodesic from a_0 to a_1 passes through σ in the positive direction, -1 if it passes through σ in the negative direction, and 0 if it does not pass through σ. The relation between c_ω and ν_ω is well-known in this case. The formula for the canonical ω representing $[\omega]$ when ν_ω is bounded, is due to Teitelbaum [T]. Let us quickly recover it.

Start with a measure μ on $\mathcal{A}^2$. Introduce the basis $e_0 = (1, 0)$ and $e_1 = (0, 1)$ of $V_K = K^2$. The corresponding measure ν on $\mathbb{P}^1(K) = \mathcal{B}^{\min}_{k-1}$ is given by

$$\nu(U) = \mu\left\{(e_0 g, e_1 g) \middle|\, g^{-1}(0) \in U\right\} - \mu\left\{(e_0 g, e_1 g) \middle|\, g^{-1}(\infty) \in U\right\},$$

where g denotes an element of $\mathrm{PGL}_2(K)$. Note that if μ is degenerate, $\nu = 0$, and that ν is harmonic, which in this case ($k = 1$) means that it is of total mass 0. An easy computation yields

$$\omega(\mu) = \int_{\mathcal{A}^2} \omega(S) d\mu(S) = \int_{\mathbb{P}^1(K)} \frac{dz}{z - \zeta} d\nu(\zeta).$$

2. The proofs

2.1. THE LEMMA OF DELIGNE-SCHNEIDER-STUHLER. Let $\mathcal{F}^{(k)}$ denote the profinite simplicial set of flags of subspaces of V_K whose dimensions do not exceed k ([S-S], p. 66):

$$\mathcal{F}^{(k)}_{r-1} = \{V_K \supsetneq W_1 \supsetneq \cdots \supsetneq W_r;\ 1 \leq \dim W_i \leq k\}, \tag{2.1}$$

and let $\mathcal{NF}_{\cdot}^{(k)}$ be the subset of non-degenerate flags (i.e., flags where W_i properly contains W_{i+1}). Note that $\mathcal{NF}_{r-1}^{(k)}$ is a subset of $\mathcal{F}_{r-1}^{(k)}$ (if we relax the assumption of non-degeneracy), but also of $\mathcal{B}_{r-1}$ (if we relax the restriction on the dimensions of the W_i), and as such

$$\mathcal{NF}_{k-1}^{(k)} = \mathcal{B}_{k-1}^{\min}. \tag{2.2}$$

As explained in [S-S], p. 66, the complex

$$0 \to \mathbb{Z} \to L(\mathcal{F}_{\cdot}^{(k)}) \tag{2.3}$$

computes the reduced cohomology of $|\mathcal{F}_{\cdot}^{(k)}|$ with coefficients in $\mathbb{Z}$, and since it is homotopically equivalent to the complex

$$0 \to \mathbb{Z} \to L(\mathcal{NF}_{\cdot}^{(k)}) \tag{2.4}$$

(loc. cit. p. 69), we have

$$\tilde{H}^{k-1}(|\mathcal{F}_{\cdot}^{(k)}|, \mathbb{Z}) = L(\mathcal{B}_{k-1}^{\min})/\partial^* L(\mathcal{NF}_{k-2}^{(k)}), \tag{2.5}$$

with the understanding that if $k = 1$ the denominator is replaced by $\mathbb{Z}$.

Similarly, the complex

$$0 \to \mathbb{Z} \to L(Y_{\cdot}^{(k)}) \tag{2.6}$$

computes the cohomology of $|Y_{\cdot}^{(k)}|$. If we set $Y_r = \mathcal{A}^{r+1}$ (without any condition on the dimension of the Span), $Y_{\cdot}$ is acyclic, and of course $Y_r = Y_r^{(k)}$ for $r \leq k-1$. It follows that

$$\begin{aligned}\tilde{H}^{k-1}(|Y_{\cdot}^{(k)}|, \mathbb{Z}) &= \operatorname{Ker}(L(Y_{k-1}^{(k)}) \to L(Y_k^{(k)}))/\operatorname{Im}(L(Y_{k-2}^{(k)}) \to L(Y_{k-1}^{(k)})) \\ &= \operatorname{Ker}(L(Y_{k-1}^{(k)}) \to L(Y_k^{(k)}))/\operatorname{Ker}(L(Y_{k-1}^{(k)}) \to L(Y_k)) \\ &\simeq \left\{ f \in L(Y_k);\ f|_{Y_k^{(k)}} = 0 \text{ and } \partial^* f = 0 \right\},\end{aligned} \tag{2.7}$$

where the last isomorphism is induced by ∂^*, but this is just the subgroup that we denoted by Λ_k'.

LEMMA 2.1: *For every $k \geq 1$ there exists a G-equivariant isomorphism*

$$L(\mathcal{B}_{k-1}^{\min})/\partial^* L(\mathcal{NF}_{k-2}^{(k)}) \simeq \Lambda_k' \tag{2.8}$$

(the denominator on the left is equal to $\mathbb{Z}$ if $k = 1$), carrying $\chi_{\mathcal{B}(\sigma_0)}$, the characteristic function of the open and compact set $\mathcal{B}(\sigma_0)$, to the function

$$\lambda_{\sigma_0} \colon S \mapsto (\sigma_0, S) \tag{2.9}$$

(see Section 1.3).

Proof: Given (2.5) and (2.7), the existence of an isomorphism as above is a consequence of Proposition 5 of Section 3 of [S-S]. Let $Z^{(k)}_{\cdot,\cdot}$ be the bisimplicial profinite set introduced in [S-S]. One has a commutative diagram

(2.10)

$$
\begin{array}{ccccccccccc}
\cdot & | & \cdot & & & & & & & & \\
\uparrow & | & \uparrow & & & & & & & & \\
L(\mathcal{F}^{(k)}_{k-1}) & |\to & L(Z^{(k)}_{k-1,0}) & \to & \cdot & & & & & & \\
\uparrow & | & \uparrow & & \uparrow & & & & & & \\
\cdots & | & \cdots & & L(Z^{(k)}_{k-2,1}) & \to & \cdot & & & & \\
\uparrow & | & \uparrow & & & & & & & & \\
L(\mathcal{F}^{(k)}_{1}) & |\to & L(Z^{(k)}_{1,0}) & \to & \cdot & & \cdots & & \cdot & & \\
\uparrow & | & \uparrow d' & & \uparrow & & & & \uparrow & & \\
L(\mathcal{F}^{(k)}_{0}) & |\to & L(Z^{(k)}_{0,0}) & \overset{d''}{\to} & L(Z^{(k)}_{0,1}) & \to & \cdots & \to & L(Z^{(k)}_{0,k-1}) & \to & \cdot \\
\uparrow & | & \text{--}\uparrow\text{--} & \text{--} & \text{--}\uparrow\text{--} & \text{--} & \text{--} & \text{--} & \text{--}\uparrow\text{--} & \text{--} & \text{----} \\
\mathbb{Z} & \to & L(Y^{(k)}_{0}) & \to & L(Y^{(k)}_{1}) & \to & \cdots & \to & L(Y^{(k)}_{k-1}) & \to & \cdot
\end{array}
$$

Here the left column and the bottom row (which are separated from the double complex of the $Z^{(k)}_{\cdot,\cdot}$ by dashes) are the complexes used to compute the reduced cohomologies $\tilde{H}^s(|\mathcal{F}^{(k)}_{\cdot}|, \mathbb{Z})$ and $\tilde{H}^s(|Y^{(k)}_{\cdot}|, \mathbb{Z})$. These reduced cohomologies vanish for $s < k-1$ and we are interested in their value for $s = k-1$, which is why we wrote the kth diagonal in the double complex (corresponding to $Z^{(k)}_{r,s}$ with $r + s = k - 1$) in full.

The *augmented* rows and columns of the double complex are acyclic. This is the content of Lemma 3, p. 67 of [S-S], and is straightforward.

To compute the effect of the isomorphism

$$
\tilde{H}^{k-1}(|\mathcal{F}^{(k)}_{\cdot}|, \mathbb{Z}) \simeq \tilde{H}^{k-1}(|Y^{(k)}_{\cdot}|, \mathbb{Z}) \tag{2.11}
$$

on the class of $\chi_{\mathcal{B}(\sigma_0)} \in L(\mathcal{F}^{(k)}_{k-1})$ we have to "go down the stairs" along the kth diagonal in the diagram until we find the corresponding function χ_Y in $L(Y^{(k)}_{k-1})$ whose class $[\chi_Y] \in \tilde{H}^{k-1}(|Y^{(k)}_{\cdot}|, \mathbb{Z})$ is the desired class. We then have to show (recall 2.7) that $\partial^* \chi_Y \in L(Y_k)$ (which of course vanishes on $Y^{(k)}_k$ and is therefore in Λ'_k) is the function $S \longmapsto (\sigma_0, S)$.

Let $\chi_0, \ldots, \chi_{k-1}$, $\chi_i \in L(Z^{(k)}_{k-1-i,i})$, be the functions constructed in the transition from $\chi_{\mathcal{B}(\sigma_0)}$ to χ_Y. Of course, there is some ambiguity in the process, as the resulting element of $L(Y^{(k)}_{k-1})$ is only well-defined up to coboundaries, but we shall select nice candidates for the χ_i.

The first step gives that χ_0 is the characteristic function of the set

$$Q_{k-1,0}=\left\{\begin{array}{c}(W_1\supset\cdots\supset W_k;Ka_k)\in Z^{(k)}_{k-1,0}|\\(W_1\supset\cdots\supset W_k=Ka_k)\text{ is compatible with }\sigma_0\end{array}\right\}. \tag{2.12}$$

Here Ka_k represents a *line* in V_K (a hyperplane H_{a_k} in V_K^*) so a_k can be normalized to lie in $L_0-\pi L_0$, and then the condition that it spans W_k means that it lies in $L_k-\pi L_0$, i.e., $\iota_{\sigma_0}(a_k)=k$. A simple computation reveals that $\chi_0=d'\psi_0$ is the coboundary (in the vertical direction) of the characteristic function ψ_0 of

$$P_{k-2,0}=\left\{\begin{array}{c}(W_1\supset\cdots\supset W_{k-1};Ka_k)\in Z^{(k)}_{k-2,0}|\\(W_1\supset\cdots\supset W_{k-1}\supset Ka_k)\text{ is compatible with }\sigma_0\end{array}\right\}. \tag{2.13}$$

Taking now the differential of ψ_0 in the horizontal direction, we see that $\chi_1=d''\psi_0$ is the *anti-symmetrization* of the characteristic function of

$$Q_{k-2,1}=\left\{\begin{array}{c}(W_1\supset\cdots\supset W_{k-1};Ka_{k-1},Ka_k)\in Z^{(k)}_{k-2,1}|\\(W_1\supset\cdots\supset W_{k-1}=Ka_{k-1}+Ka_k\supset Ka_k)\text{ is compatible with }\sigma_0\end{array}\right\}. \tag{2.14}$$

Here we use the following terminology. The symmetric group S_m acts on $Z^{(k)}_{k-m,m-1}$ by permuting the m hyperplanes. It therefore acts on $L(Z^{(k)}_{k-m,m-1})$ and the anti-symmetrization of $f\in L(Z^{(k)}_{k-m,m-1})$ is the function f^{alt} defined by

$$f^{alt}=\sum_{\pi\in S_m}\operatorname{sgn}(\pi)\cdot{}^{\pi}f. \tag{2.15}$$

Repeating the same argument, χ_1 is the coboundary (in the vertical direction) of the anti-symmetrization of the characteristic function of

$$P_{k-3,1}=\left\{\begin{array}{c}(W_1\supset\cdots\supset W_{k-2};Ka_{k-1},Ka_k)\in Z^{(k)}_{k-3,1}|\\(W_1\supset\cdots\supset W_{k-2}\supset Ka_{k-1}+Ka_k\supset Ka_k)\text{ is compatible with }\sigma_0\end{array}\right\} \tag{2.16}$$

and the (horizontal) differential of this function is χ_2, the anti-symmetrization of the characteristic function of

$$Q_{k-3,2}=\left\{\begin{array}{c}(W_1\supset\cdots\supset W_{k-2};Ka_{k-2},Ka_{k-1},Ka_k)\in Z^{(k)}_{k-3,2}|\\(W_1\supset\cdots\supset W_{k-2}=Ka_{k-2}+Ka_{k-1}+Ka_k\supset\cdots\supset Ka_k)\\\text{is compatible with }\sigma_0\end{array}\right\}. \tag{2.17}$$

At the end we arrive at χ_{k-1}, the anti-symmetrization of the characteristic function of the set

$$Q_{0,k-1}=\left\{\begin{array}{c}(W_1;Ka_1,\ldots,Ka_k)\in Z^{(k)}_{0,k-1}|\\(W_1=Ka_1+\cdots+Ka_k\supset\cdots\supset Ka_k)\text{ is compatible with }\sigma_0\end{array}\right\}. \tag{2.18}$$

We therefore see that χ_Y is the anti-symmetrization of the characteristic function of all those

$$(Ka_1, \ldots, Ka_k) \tag{2.19}$$

for which $\iota_{\sigma_0}(a_j) = j$ for each $1 \leq j \leq k$.

Finally, it is easy to see that for $S = (a_0, a_1, \ldots, a_k)$,

$$\partial^* \chi_Y(S) = \sum_{i=0}^{k} (-1)^i \chi_Y(a_0, \ldots, \hat{a}_i, \ldots, a_k) = (\sigma_0, S). \tag{2.20}$$

This concludes the proof of the lemma. ∎

Corollary 2.2: *Let $\Lambda_k \subset L(Y_k)$ be the $\mathbb{Z}$-linear span of the functions $\lambda_\sigma = (\sigma, -)$. Then $\Lambda_k = \Lambda'_k$.*

Proof: One clearly has $\Lambda_k \subset \Lambda'_k \subset L(Y_k)$. Note also that $L(Y_k)/\Lambda'_k$ is torsion-free, so for any field F

$$\Lambda'_k \otimes F \subset L(Y_k) \otimes F, \tag{2.21}$$

and $\mathrm{Hom}(\Lambda'_k, F) = \mathcal{D}(Y_k, F)/\mathcal{D}(Y_k, F)_{\mathrm{deg}}$. To prove the corollary it is enough to prove that for any field F, $\Lambda_k \otimes F = \Lambda'_k \otimes F$ (indeed it is enough to check this for $F = \mathbb{Q}$ or $\mathbb{F}_l$). If this were not the case, there would have been a non-zero $\mu \in \mathrm{Hom}(\Lambda'_k, F) = \mathcal{D}(Y_k, F)/\mathcal{D}(Y_k, F)_{\mathrm{deg}}$ vanishing on Λ_k, and in particular on the functions λ_σ. Let $\nu \in \mathcal{D}(\mathcal{B}^{\min}_{k-1}, F)_{har}$ be the distribution on $\mathcal{B}^{\min}_{k-1}$ which corresponds to μ under the lemma. Then

$$\nu(\mathcal{B}(\sigma)) = \mu(\lambda_\sigma) = 0 \tag{2.22}$$

for every σ of minimal type. As the $\mathcal{B}(\sigma)$ form a basis for the topology of $\mathcal{B}^{\min}_{k-1}$, $\nu = 0$, hence $\mu = 0$ as well. This contradiction shows that the $\mathbb{Z}$-span of the λ_σ is precisely the kernel of (ι^*, ∂^*), as desired. ∎

2.2. Proof of Theorem 1.1. In [dS] we showed that C^k_{har} is the algebraic dual of $\Lambda_k \otimes K$, and that

$$\langle \lambda_\sigma, c_{\omega(S)} \rangle = c_{\omega(S)}(\sigma) = (\sigma, S). \tag{2.23}$$

From the lemma and its corollary we get, upon dualizing, the equalities

$$\mathcal{D}(\mathcal{B}^{\min}_{k-1}, K)_{har} \simeq \mathcal{D}(\mathcal{A}^{k+1}, K)/\mathcal{D}(\mathcal{A}^{k+1}, K)_{\mathrm{deg}} \simeq C^k_{har} \tag{2.24}$$

as in (i), (ii) and (iii) of Theorem 1.1. See the commutative diagram below, in which Ψ is dual to the identity $\Lambda_k = \Lambda'_k$. Since two sides of the diagram are isomorphisms, so is the third, Φ.

$$\mathcal{D}(\mathcal{B}^{\min}_{k-1}, K)_{har} \overset{[2.1]}{\simeq} \mathcal{D}(\mathcal{A}^{k+1}, K)/\mathcal{D}(\mathcal{A}^{k+1}, K)_{\deg} \quad \begin{array}{l} \xrightarrow{[I-S]\Phi} H^k_{dR}(\mathfrak{X}/K) \\ \qquad\qquad \downarrow \boxed{\omega \mapsto c_\omega} \\ \xrightarrow{\Psi} C^k_{har} \end{array}$$

On the other hand, the Main Theorem of [I-S], together with Lemma 2.1, imply the formula

$$c^{SS}_{\omega(S)}(\chi_{B_I}) = (\sigma_0, S), \tag{2.25}$$

which gives part (iv) of the theorem.

2.3. PROOF OF THEOREM 1.2. If instead of taking $\mathrm{Hom}(-, K)$ in the proof of Theorem 1.1, we take $\mathrm{Hom}(-, \mathcal{O}_K)$, one gets the first part of Theorem 1.2 (the equality of the norms on the bounded cohomology in the three interpretations). As for the second, simply note, as in [I-S], that the isomorphism

$$\Phi\colon M(\mathcal{A}^{k+1}, K)/M(\mathcal{A}^{k+1}, K)_{\deg} \simeq H^k_{dR}(\mathfrak{X}/K)_{bnd} \tag{2.26}$$

factors through Ω^k_{bnd} (see 1.15).

3. The cohomology of $\Gamma\backslash\mathfrak{X}$

3.1. A HODGE-LIKE DECOMPOSITION OF THE COHOMOLOGY. Let Γ be a discrete cocompact subgroup of G. For simplicity make the assumption that if $1 \neq \gamma \in \Gamma$, then for every vertex v of $\mathcal{T}$, the distance between v and γv is at least 3. This assumption is needed in order to make $\Gamma\backslash\mathcal{T}$ into a simplicial complex, and can be met by passing to a subgroup of finite index.* By a theorem of Mustafin [Mus], $\Gamma\backslash\mathfrak{X}$ is the rigid analytic space associated to a unique smooth projective variety X_Γ, defined over K. Let $\pi_\Gamma\colon \mathfrak{X} \to X_\Gamma$ be the canonical projection. For

* Our condition guarantees that if $(v_0, \ldots, v_r) \in \mathcal{T}_r$, and $\gamma_i \in \Gamma$, then the $\gamma_i v_i$ are distinct, and if $(\gamma_0 v_0, \ldots, \gamma_r v_r) \in \mathcal{T}_r$, then all the γ_i are equal. Defining $(\Gamma\backslash\mathcal{T})_r = \Gamma\backslash\mathcal{T}_r$ makes $\Gamma\backslash\mathcal{T}$ a simplicial complex, and $|\Gamma\backslash\mathcal{T}| = \Gamma\backslash|\mathcal{T}|$.

$\tau \in \mathcal{T}_r$ let $\mathrm{St}(\tau)$ (the star of τ) be the union of all the (open) simplices of $|\mathcal{T}|$ containing τ. Define

$$X(\tau) := r^{-1}(\mathrm{St}(\tau)) = \bigcap_{v \in \tau} X(v) \tag{3.1}$$

and $X_\Gamma(\tau) := \pi_\Gamma(X(\tau))$. Then $\mathcal{U} = \{X(v);\ v \in \mathcal{T}_0\}$ and $\mathcal{U}_\Gamma = \{X_\Gamma(v);\ v \in \Gamma\backslash\mathcal{T}_0\}$ are admissible coverings of $\mathfrak{X}$ and X_Γ, respectively, whose nerves are just $\mathcal{T}$ and $\Gamma\backslash\mathcal{T}$.

There are two spectral sequences abutting to the de-Rham cohomology $H^n_{dR}(X_\Gamma)$. (Note that by GAGA, analytic and algebraic de-Rham cohomology coincide.) The first is the *covering spectral sequence*

$$'E_2^{r,s} = H^r(\Gamma, H^s_{dR}(\mathfrak{X})) \Rightarrow H^n_{dR}(X_\Gamma). \tag{3.2}$$

The associated filtration is denoted by $F^{\cdot}_\Gamma H^n_{dR}(X_\Gamma)$. The second is the *Hodge-to-de-Rham spectral sequence*

$$''E_1^{r,s} = H^s(X_\Gamma, \Omega^r) \Rightarrow H^n_{dR}(X_\Gamma), \tag{3.3}$$

and the associated filtration is the Hodge filtration $F^{\cdot}_{dR} H^n_{dR}(X_\Gamma)$. Below, we shall construct them as the two spectral sequences attached to a certain double complex (see also [dS], Section 9).

Our purpose is to show how the isomorphism $\Omega^s_{bnd} \simeq H^s_{dR}(\mathfrak{X})_{bnd} \simeq C^s_{bnd}$ and the results of [dS] yield the following theorem.

THEOREM 3.1: (i) *The covering spectral sequence degenerates at* $'E_2$.

(ii) *The Hodge-to-de-Rham spectral sequence degenerates at* $''E_1$.

(iii) *The two filtrations are opposite to each other: For every* $0 \le n \le d$, *and every* p,

$$H^n_{dR}(X_\Gamma) = F_\Gamma^{n+1-p} H^n \oplus F^p_{dR} H^n. \tag{3.4}$$

As a corollary, if we put, for $p + q = n$,

$$H^{q,p} = F^q_\Gamma H^n \cap F^p_{dR} H^n, \tag{3.5}$$

we have a Hodge-like direct sum decomposition

$$H^n = \oplus H^{q,p}. \tag{3.6}$$

The three statements of the theorem are not new. Part (i) was proved by Schneider in [S]. Part (ii) is classical, and is usually proved by complex Hodge theory (the theory of harmonic forms on Kähler manifolds). Our substitute for the complex theory is harmonic analysis on the building, and, as far as we know, this constitutes a new proof (for p-adically uniformized varieties). Part (iii) was conjectured by Schneider in [S], and proved by Iovita and Spiess in [I-S]. Our proof is closer in spirit to Section 9 of [dS], but the key idea, namely the existence of a Γ-module Ω^s_{bnd} mapping isomorphically onto the bounded cohomology, is the same.

3.2. Generalities on double complexes. We recall the two spectral sequences attached to a double complex. See [Go] I.4.8 for the details. Let $K^{r,s}$ be a double complex of vector spaces vanishing for $r < 0$ or $s < 0$. We denote the differentials of the complex by

$$d'\colon K^{r,s} \to K^{r+1,s}, \quad d''\colon K^{r,s} \to K^{r,s+1}, \quad d'd'' + d''d' = 0. \tag{3.7}$$

The total complex is

$$\mathrm{tot}(K)^n = \bigoplus_{r+s=n} K^{r,s}, \quad D = d' + d''. \tag{3.8}$$

There are two spectral sequences abutting to the homology $H^n = h^n(\mathrm{tot}(K)^\cdot, D)$. The *first* spectral sequence starts with

$${}'E_1^{r,s} = h^s(K^{r,\cdot}, d'') \tag{3.9}$$

and ${}'d_1^{r,s}\colon {}'E_1^{r,s} \to {}'E_1^{r+1,s}$ is derived from d'. The induced filtration on the abuttment is denoted ${}'F^\cdot H^n$. The *second* spectral sequence is obtained by reversing the roles of the indices. Thus

$${}''E_1^{r,s} = h^s(K^{\cdot,r}, d'), \tag{3.10}$$

where ${}''d_1^{r,s}$ is derived from d'', and the induced filtration is denoted by ${}''F^\cdot H^n$.

A map of complexes induces maps between the (first or second) spectral sequences, and maps on the abuttments, compatible with both filtrations.

3.3. A trivial but useful example. Suppose that $d'' = 0$ throughout. Then putting $H^{r,s} = h^r(K^{\cdot,s}, d')$,

$$H^n = \bigoplus_{r+s=n} H^{r,s}. \tag{3.11}$$

The *first* spectral sequence gives ${}'E_1^{r,s} = K^{r,s}$, and degenerates at ${}'E_2^{r,s} = H^{r,s}$ (i.e., ${}'d_2^{r,s} = {}'d_3^{r,s} = \cdots = 0$). For the corresponding filtration we find

$${}'F^q H^n = \bigoplus_{r\geq q} H^{r,n-r}. \tag{3.12}$$

On the other hand, the *second* spectral sequence degenerates at $''E_1^{r,s} = H^{s,r}$ and

$$''F^pH^n = \bigoplus_{r\geq p} H^{n-r,r}. \tag{3.13}$$

In particular, the two filtrations are opposite to each other, and if $p+q=n$,

$$'F^qH^n \cap {}''F^pH^n = H^{q,p}. \tag{3.14}$$

LEMMA 3.2: *Let $\varphi\colon \tilde{K}^{r,s} \to K^{r,s}$ be a map of double complexes. Assume that for $\tilde{K}^{r,s}$, we have $\tilde{d}''=0$ throughout.*

(i) *Assume that the maps $'E_2\varphi\colon {}'\tilde{E}_2^{r,s} \to {}'E_2^{r,s}$ are isomorphisms. Let $\tilde{H}^n$ and H^n be the homologies of the total complexes. Then the induced map $H\varphi\colon \tilde{H}^n \to H^n$ is an isomorphism for every n, under which $'\tilde{F}^\cdot\tilde{H}^n$ and $'F^\cdot H^n$ agree (i.e., map isomorphically to each other), and the spectral sequence $'E_2^{r,s} \Rightarrow H^n$ degenerates at $'E_2$.*

(ii) *If, furthermore, $\infty > \dim {}''\tilde{E}_1^{r,s} \geq \dim {}''E_1^{r,s}$ for every r and s, then $''E_1\varphi\colon {}''\tilde{E}_1^{r,s} \to {}''E_1^{r,s}$ is an isomorphism, the filtrations $''\tilde{F}^\cdot\tilde{H}^n$ and $''F^\cdot H^n$ agree, and the spectral sequence $''E_1^{r,s} \Rightarrow H^n$ degenerates at $''E_1$.*

Proof: (i) Since the first spectral sequence of the double complex $\tilde{K}^{\cdot,\cdot}$ degenerates at $'\tilde{E}_2$, we get, under the assumption that $'E_2\varphi$ is an isomorphism, that the same is true for the complex $K^{\cdot,\cdot}$. Furthermore, the map $H\varphi$ on the abuttments induces isomorphisms on the graded pieces with respect to the first filtration, and is therefore an isomorphism of *filtered* vector spaces.

(ii) Since $''\tilde{E}_1^{r,s} = {}''\tilde{E}_\infty^{r,s}$, a dimension count shows that we must have $''E_1^{r,s} = {}''E_\infty^{r,s}$ as well, so the second spectral sequence of the complex $K^{\cdot,\cdot}$ degenerates at $''E_1$ too. The same dimension count shows that we must have an equality $\dim {}''\tilde{E}_1^{r,s} = \dim {}''E_1^{r,s}$. The map $H\varphi$ is an isomorphism of vector spaces, compatible with the second filtration on $\tilde{H}^n$ and H^n. As we have just remarked, the graded pieces with respect to the second filtration have the same dimensions. Thus $H\varphi$ is an isomorphism of *filtered* vector spaces also with respect to the *second* filtration. ∎

COROLLARY 3.3: *Under the conditions* (i) *and* (ii) *of the lemma, the filtrations $'F^\cdot$ and $''F^\cdot$ on H^n are opposite to each other.*

3.4. THE DOUBLE COMPLEX $K^{\cdot,\cdot}$. We repeat some of the arguments of [dS], Section 9. Introduce the double complex

$$K^{r,s} = C^r(\mathcal{U}_\Gamma, \Omega^s) = C^r(\mathcal{U}, \Omega^s)^\Gamma \tag{3.15}$$

of Čech cochains for the cover $\mathcal{U}_\Gamma$ with values in the presheaf of rigid analytic forms. Thus an element of $C^r(\mathcal{U}, \Omega^s)$ is a map f which assigns to every $\tau \in \widehat{\mathcal{T}}_r$ an element $f(\tau) \in \Omega^s(X(\tau))$. The action of $\gamma \in \Gamma$ is given as usual by the rule $(\gamma f)(\tau) = \gamma(f(\gamma^{-1}\tau))$, and $K^{r,s}$ is by definition the subspace of Γ-invariants. The differentials are $d' = \delta$ (Čech differntiation) and $d'' = (-1)^r d$. The total complex $\mathrm{tot}(K)^{\cdot}$ computes the Čech hyper-cohomology $\check{H}^n(\mathcal{U}_\Gamma, \Omega^{\cdot})$. Since the open sets of the cover and all their intersections are $\Omega^{\cdot}$-acyclic, we find that the homology of the total complex is

$$H^n = H^n(X_\Gamma, \Omega^{\cdot}) = H^n_{dR}(X_\Gamma). \tag{3.16}$$

Let us compute the two spectral sequences attached to this double complex. The first is

$${}'E_1^{r,s} = C^r(\mathcal{U}_\Gamma, \underline{H}^s_{dR}). \tag{3.17}$$

Here $\underline{H}^s_{dR}$ is the presheaf on $\mathcal{U}$ of (rigid) de-Rham cohomology. As explained in [dS], it is isomorphic to the presheaf denoted there by $\underline{A}^s$. Now the *acyclicity theorem* ([dS], Theorem 5.7) claims that the Čech complex

$$0 \to C^s_{har} \to C^{\cdot}(\mathcal{U}, \underline{A}^s) \tag{3.18}$$

is an acyclic resolution of the Γ-module of harmonic s-cochains on the building. We therefore find that

$${}'E_2^{r,s} = \check{H}^r(\mathcal{U}_\Gamma, \underline{H}^s_{dR}) = h^r(C^{\cdot}(\mathcal{U}, \underline{A}^s)^\Gamma, \delta) = H^r(\Gamma, C^s_{har}). \tag{3.19}$$

Thus the first spectral sequence is nothing but the *covering spectral sequence*

$$H^r(\Gamma, H^s_{dR}(\mathfrak{X})) \Rightarrow H^n_{dR}(X_\Gamma). \tag{3.20}$$

The second spectral sequence is, by definition, the Hodge-to-de-Rham spectral sequence

$${}''E_1^{r,s} = \check{H}^s(\mathcal{U}_\Gamma, \Omega^r) = H^s(X_\Gamma, \Omega^r) \Rightarrow H^n_{dR}(X_\Gamma). \tag{3.21}$$

3.5. THE DOUBLE COMPLEX $\tilde{K}^{\cdot,\cdot}$, AND THE END OF THE PROOF. Consider now the subcomplex $\tilde{K}^{r,s} \subset K^{r,s}$,

$$\tilde{K}^{r,s} = C^r(\mathcal{U}_\Gamma, \Omega^s_{bnd}) = C^r(\mathcal{U}, \Omega^s_{bnd})^\Gamma, \tag{3.22}$$

of Čech cocycles with values in the locally constant system of *global bounded s-forms*. Thus an element of $C^r(\mathcal{U}, \Omega^s_{bnd})$ is a function f which assigns to each $\tau \in \hat{\mathcal{T}}_r$ an s-form $f(\tau) \in \Omega^s_{bnd} \subset \Omega^s(X(\tau))$, and f is Γ-invariant if $f(\gamma(\tau)) = \gamma f(\tau)$ for every $\gamma \in \Gamma$. Since bounded forms are closed, $d'' = 0$ on $\tilde{K}^{r,s}$.

If we take for φ the inclusion of $\tilde{K}^{r,s}$ in $K^{r,s}$, we find ourselves in the situation of Lemma 3.2. In order to apply it, one has to check conditions (i) and (ii). Recalling the isomorphism $\Omega^s_{bnd} \simeq C^s_{bnd}$ we see that

$$'\tilde{E}_2^{r,s} = \check{H}^r(\mathcal{U}_\Gamma, C^s_{bnd}) \simeq H^r(\Gamma, C^s_{bnd}). \tag{3.23}$$

The last isomorphism stems from the fact that $|\mathcal{T}|$ is a contractible space on which Γ acts, and the nerve of $\mathcal{U}_\Gamma$ is $\Gamma\backslash\mathcal{T}$. Condition (i) boils down now to the following lemma.

LEMMA 3.4: *The inclusion of C^s_{bnd} in C^s_{har} induces an isomorphism in group cohomology*

$$H^r(\Gamma, C^s_{bnd}) \simeq H^r(\Gamma, C^s_{har}). \tag{3.24}$$

Proof: The proof is based on the *acyclicity theorem* ([dS], Theorem 5.7). The Γ-module $C^s_{bnd}(\mathcal{O}_K)$ of the integral-valued harmonic s-cochains admits an acyclic resolution

$$0 \to C^s_{bnd}(\mathcal{O}_K) \to C^\cdot(\mathcal{U}, \underline{A}^s(\mathcal{O}_K)) \tag{3.25}$$

similar to the one for C^s_{har}. To deduce it from [dS] observe that the arguments there yield the wanted version with integral coefficients because $C^s_{bnd}(\mathcal{O}_K) = \mathrm{Hom}(\Lambda_k, \mathcal{O}_K)$, and because theorem 5.3 of [dS] holds for Λ_k with coefficients in $\mathbb{Z}$, as we take it to be in this work, even before we extend scalars to K.

We therefore have

$$H^r(\Gamma, C^s_{bnd}(\mathcal{O}_K)) = h^r(C^\cdot(\mathcal{U}, \underline{A}^s(\mathcal{O}_K))^\Gamma). \tag{3.26}$$

Since Γ is finitely generated and $\mathcal{U}_\Gamma$ is finite,

$$\begin{aligned} H^r(\Gamma, C^s_{bnd}) &= h^r(C^\cdot(\mathcal{U}, \underline{A}^s(\mathcal{O}_K))^\Gamma \otimes_{\mathcal{O}_K} K) \\ &= h^r(C^\cdot(\mathcal{U}, \underline{A}^s(K))^\Gamma) = H^r(\Gamma, C^s_{har}). \end{aligned} \tag{3.27}$$ ∎

The second point to check is condition (ii) of Lemma 3.2. But we have already seen that $''E_1^{r,s} = H^s(X_\Gamma, \Omega^r)$, and from the last lemma $''\tilde{E}_1^{r,s} = H^s(\Gamma, C_{har}^r) \simeq H^s(\Gamma, H_{dR}^r(\mathfrak{X}))$. The dimensions of these two pieces have been computed by Schneider in [S]:

$$\dim H^s(X_\Gamma, \Omega^r) = \dim H^s(\Gamma, H_{dR}^r(\mathfrak{X})) = \mu(\Gamma)\delta_{r+s,d} + \delta_{r,s} \tag{3.28}$$

where $\mu(\Gamma) = \dim H^d(\Gamma, K)$.

Lemma 3.2 and Corollary 3.3 now conclude the proof of Theorem 3.1. Incidentally, note that the isomorphism $''\tilde{E}_1^{r,s} \simeq {}''E_1^{r,s}$ is the isomorphism

$$gr_\Gamma^s H^n = H^{s,r} = gr_{dR}^r H^n. \tag{3.29}$$

References

[dS] E. de Shalit, *Residues on buildings and de Rham cohomology of p-adic symmetric domains*, Duke Mathematical Journal **106** (2000), 123–191.

[Go] R. Godement, *Topologie algebrique et théorie des faisceaux*, Hermann, Paris, 1973.

[I-S] A. Iovita and M. Spiess, *Logarithmic differential forms on p-adic symmetric spaces*, preprint, January 2000.

[Mus] G. A. Mustafin, *Nonarchimedean uniformization*, Mathematics of the USSR—Sbornik **34** (1978), 187–214.

[S] P. Schneider, *The cohomology of local systems on p-adically uniformized varieties*, Mathematische Annalen **293** (1992), 623–650.

[S-S] P. Schneider and U. Stuhler, *The cohomology of p-adic symmetric spaces*, Inventiones Mathematicae **105** (1991), 47–122.

[S-T] P. Schneider and J. Teitelbaum, *An integral transform for p-adic symmetric spaces*, Duke Mathematical Journal **86** (1997), 391–433.

[T] J. Teitelbaum, *Values of p-adic L-functions and a p-adic Poisson kernel*, Inventiones Mathematicae **101** (1990), 395–410.